Flavors of bicycle mathematics

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Abstract

This talk concerns a naive model of bicycle motion: a bicycle is a segment of fixed length that can move so that the velocity of the rear end is always aligned with the segment. Surprisingly, this simple model is quite rich and has connections with several areas of research, including completely integrable systems. Here is a sampler of problems that I hope to touch upon:

1) The trajectory of the front wheel and the initial position of the bicycle uniquely determine its motion and its terminal position; the monodromy map sending the initial position to the terminal one arises. This mapping is a Moebius transformation, a remarkable fact that has various geometrical and dynamical consequences.

2) The rear wheel track and a choice of the direction of motion uniquely determine the front wheel track; changing the direction to the opposite, yields another front track. These two front tracks are related by the bicycle (Backlund, Darboux) correspondence, which defines a discrete time dynamical system on the space of curves. This system is completely integrable and it is closely related with another, well studied, completely integrable dynamical system, the filament (a.k.a binormal, smoke ring, local induction) equation.

3) Given the rear and front tracks of a bicycle, can one tell which way the bicycle went? Usually, one can, but sometimes one cannot. The description of these ambiguous tire tracks is an open problem, intimately related with Ulam's problem in flotation theory (in dimension two): is the round ball the only body that floats in equilibrium in all positions? This problem is also related to the motion of a charge in a magnetic field of a special kind. It turns out that the known solutions are solitons of the planar version of the filament equation.