Representations of virtual braid groups to rook algebras and virtual links invariants

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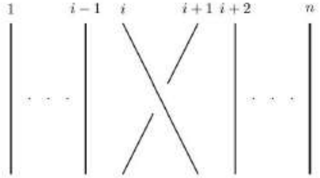
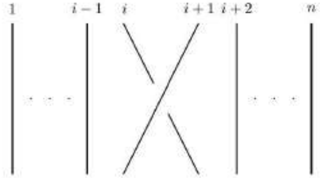
Braid group on n strands, denoted by $\mathcal{B}_n$, is a group generated by $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \text{ if } |i - j| > 1,$$
$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i, \quad \text{for } i = 1, \ldots, n-2.$$

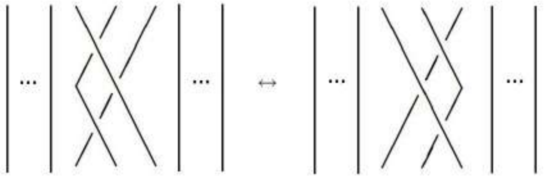
Fix the points $P_i = (i, 1)$ and $Q_i = (i, 0)$ in $\mathbb{R}^2$ for $i = 1, 2, \ldots, n$. For braid word ω, presented braid $\beta \in \mathcal{B}_n$, we connect P_i and Q_j by drawing following diagrams





for generators σ_i and σ_i^{-1}.
The result is called the diagram of braid β

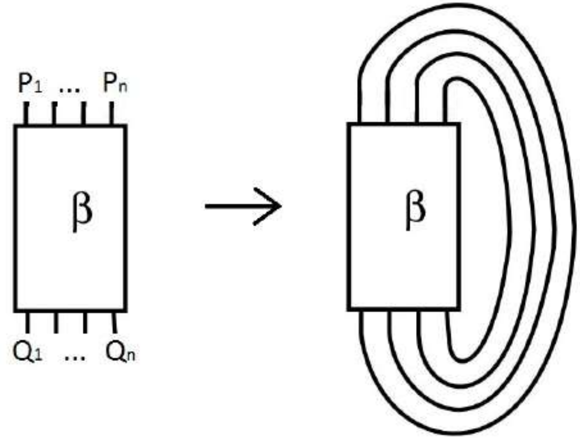
Relations of braid groups correspond to plane isotopies and Reidemeister moves 2 and 3.



The geometrical interpretation of relation $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$.

The set of all braid diagrams up to isotopies an and Reidemeister moves form a group, isomorphic to braid group $\mathcal{B}_n$.

Closure of the braid β is the link, that can be obtained of geometric representative of braid β by identifying Q_i and P_i for $i = 1, \dots, n$.



Theorem (J.Alexander)

Every link can be represented as a closed braid.

Theorem (A.Markov)

Two braids $\beta_1 \in \mathcal{B}_\mathrm{n}, \beta_2 \in \mathcal{B}_\mathrm{m}$ has the same closures if and only if beta$_2$ can be obtained from beta$_1$ by sequence of following moves or its inverses:

1. $\alpha \to {\sigma_\mathrm{i}}^{-1}\alpha\sigma_\mathrm{i}$,
2. $\alpha \to \ell(\alpha)\sigma_\mathrm{n}^{\pm 1}$,

here $\alpha, \sigma_\mathrm{i} \in \mathcal{B}_\mathrm{n}$, $\sigma_\mathrm{n} \in \mathcal{B}_{\mathrm{n}+1}$ and ℓ is a natural embedding of $\mathcal{B}_\mathrm{n}$ to $\mathcal{B}_{\mathrm{n}+1}$.

Now we can consider links as braids up to Markov moves.

Virtual braid group on n strands, denoted by $V\mathcal{B}_n$, is a group with generators:

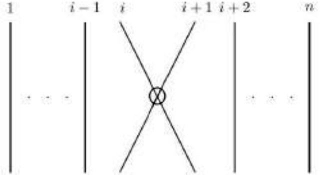
$$\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}$$

and relations:

$$\begin{gathered} \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } |i-j| > 1, \\ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i, \quad \text{for } i = 1, \dots, n-2, \end{gathered}$$

$$\begin{gathered} \rho_i^2 = e, \quad \text{for } i = 1, \dots, n-1, \\ \rho_{i+1}\rho_i\rho_{i+1} = \rho_i\rho_{i+1}\rho_i, \quad \text{for } i = 1, \dots, n-2, \\ \rho_i\rho_j = \rho_j\rho_i, \quad \text{if } |i-j| > 1, \end{gathered}$$

$$\begin{gathered} \rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1}, \quad \text{for } i = 1, \dots, n-2, \\ \sigma_i\rho_j = \rho_j\sigma_i, \quad \text{if } |i-j| > 1. \end{gathered}$$



The diagrammatic interpretation of generator ρ_{i}.

Closure of virtual braid is defined similarly as closure of classical braid.

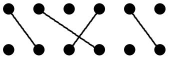
Let $\mathrm{R_n}, \mathrm{n} \geq 1$ denote a set of $\mathrm{n} \times \mathrm{n}$ matrices with entries from the set $\{0, 1\}$ having at most one 1 in each row and in each column.

Example for n = 2

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$\mathrm{R_n}$ with the standard matrix multiplication is monoid, called a rook monoid.

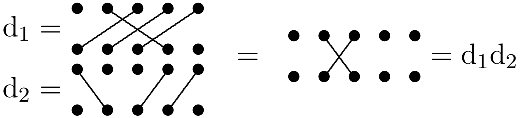
Rook diagram is a bipartite graph with n vertices in each partite, such that each vertex has degree either zero or one. We will draw one partite on the top and another on bottom of a rectangle.



$$\leftrightarrow \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There is one-to-one correspondence between rook diagrams and matrices of $\mathrm{R_n}$.

Let d_1 and d_2 be rook diagrams with the same number $2n$ of vertices. The product $d_1 d_2$ is a rook diagram with $2n$ vertices and edges, defined by the rule presented at the following picture.



Set of all diagram with this geometrical defined multiplication is monoid, isomorphic to R_n

Given diagrams d_1 and d_2, we define the tensor product, denoted $\mathrm{d}_1 \otimes \mathrm{d}_2$, to be the result of appending of d_2 to the right of d_1.

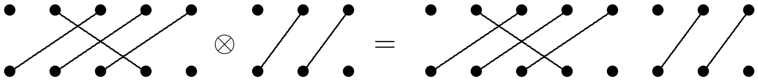


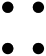
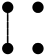
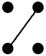
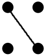
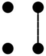
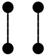
Diagram from R_n is said to be planar if it can be drawn (keeping inside of the rectangle formed by its vertices) without any crossings of edges.

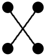
Denote by P_n the set of all planar diagrams of R_n. It is easy to see that P_n is a submonoid of R_n.

A rook algebra, denoted by $\mathbb{C}\mathrm{R_n}$, is a $\mathbb{C}$-algebra generated by $\mathrm{R_n}$.

A planar rook algebra, denoted by $\mathbb{C}\mathrm{P_n}$, is a $\mathbb{C}$-algebra generated by $\mathrm{P_n}$.

We denote elements of R_2 as following:

$d_1 =$  $d_2 =$  $d_3 =$  $d_4 =$  $d_5 =$  $d_6 =$ 

$d_7 =$ 

Define mapping $\varphi : \mathbb{B}_n \to \mathbb{C}P_n$ by the following rule:

$$\varphi(\sigma_i) = a \cdot d_{1i} + b \cdot d_{2i} + c \cdot d_{3i} + d \cdot d_{4i} + e \cdot d_{5i} + d_{6i}$$

where

$$d_{ji} = I^{\otimes i-1} \otimes d_j \otimes I^{\otimes n-i-1},$$

$a, b, c, d, e \in \mathbb{C}$ and I is the identity diagram in P_1.

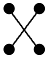
Theorem (S.Bigelow, E.Ramos, R. Yi)

Assuming $a + c + d \neq 1$ and $cd \neq 0$, a mapping of the above form is a homomorphism if and only if its coefficients are in one of the following families:

1. $b = e = -1$,
2. $a = -c - d$, $b = -1$, $e = -cd$,
3. $a = -c - d$, $b = -cd$, $e = -1$,
4. $a = 1 - c - d + cd$, $b = -cd$, $e = -1$,
5. $a = 1 - c - d + cd$, $b = -1$, $e = -cd$.

Define mapping $\psi_k : V\mathbb{B}_n \to \mathbb{C}R_n$ by the following rule:

$$\psi_k(\sigma_i) = \varphi_k(\sigma_i)$$
$$\psi_k(\rho_i) = d_{i,7}$$

$d_7 =$ 

Theorem 1

The mapping ψ_k is a representation of $V\mathcal{B}_n$ for any $k = 1, \ldots, 5$.

Example 1

Let $\psi_5^{2,3}$ be the particular case of ψ_5 for $\mathrm{c} = 2, \mathrm{d} = 3$. It is known that the braid $\beta = (\sigma_1^2\rho_1\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1)^2 \in \mathrm{V}\mathcal{B}_2$ cannot be distinguished from the trivial by the Burau presentation.
Direct computations show that

$$\psi_5^{2,3}\left((\sigma_1^2\rho_1\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1)^2\right) = \\ -\frac{2200}{9}\mathrm{d}_1 - \frac{500}{27}\mathrm{d}_2 - \frac{2450}{27}\mathrm{d}_3 + \frac{1550}{27}\mathrm{d}_4 + \frac{8000}{27}\mathrm{d}_5 + \mathrm{d}_6,$$

so $\psi_5^{2,3}$ distinguish it from the trivial braid.

$$\dim(\mathbb{C}\mathrm{P_n}) = |\mathrm{P_n}| = \sum_{\mathrm{k}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}}^2$$

For $\mathrm{n} = 1, 2, 3, 4, 5, 6$ we get $2, 6, 20, 70, 252, 924$.

$$\dim(\mathbb{C}\mathrm{R_n}) = |\mathrm{R_n}| = \sum_{\mathrm{k}=0}^{\mathrm{n}} \binom{\mathrm{n}}{\mathrm{k}}^2 \mathrm{k}!.$$

For $\mathrm{n} = 1, 2, 3, 4, 5, 6$ we get $2, 7, 34, 209, 1546, 13327$.

Let $[\ ] : \mathbb{C}\mathrm{R}_\mathrm{n} \to \mathrm{M}_\mathrm{n}(\mathbb{C})$ be a linear mapping, defined for any $\mathrm{d} \in \mathrm{R}_\mathrm{n}$ as matrix, corresponding to diagram d.

Considering coefficient c as variable, we define mapping $\phi : \mathrm{V}\mathcal{B}_\mathrm{n} \to \mathrm{GL}_\mathrm{n}(\mathbb{Z}[\mathrm{c}^{\pm 1}])$ by the following rule:

$$\phi(\sigma_\mathrm{i}) = -\frac{1}{\mathrm{c}^2}[\psi_2(\sigma_\mathrm{i})]\Big|_{\mathrm{d}=-\mathrm{c}} = \begin{pmatrix} \mathrm{I}_{\mathrm{i}-1} & & & \\ & 0 & \frac{1}{\mathrm{c}} & \\ & -\frac{1}{\mathrm{c}} & \frac{1+\mathrm{c}^2}{\mathrm{c}^2} & \\ & & & \mathrm{I}_{\mathrm{n}-\mathrm{i}-1} \end{pmatrix}$$

$$\phi(\rho_\mathrm{i}) = [\psi_2(\rho_\mathrm{i})] = \begin{pmatrix} \mathrm{I}_{\mathrm{i}-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & \mathrm{I}_{\mathrm{n}-\mathrm{i}-1} \end{pmatrix}$$

Theorem (L.Kauffman, S.Lambropoulou)

Two oriented virtual links are isotopic if and only if any two corresponding virtual braids differ by a finite sequence of braid relations $\mathrm{V}\mathcal{B}_\infty$ and the following moves or their inverses:

1. $\rho_{\mathrm{i}}\alpha\rho_{\mathrm{i}} \leftarrow \alpha \rightarrow {\sigma_{\mathrm{i}}}^{-1}\alpha\sigma_{\mathrm{i}}$,
2. $\ell(\alpha)\rho_{\mathrm{n}} \leftarrow \alpha \rightarrow \ell(\alpha)\sigma_{\mathrm{n}}^{\pm 1}$,
3. $\alpha \rightarrow \ell(\alpha)\sigma_{\mathrm{n}}^{-1}\rho_{\mathrm{n}-1}\sigma_{\mathrm{n}}$,
4. $\alpha \rightarrow \ell(\alpha)\rho_{\mathrm{n}}\rho_{\mathrm{n}-1}\sigma_{\mathrm{n}-1}\rho_{\mathrm{n}}\sigma_{\mathrm{n}-1}^{-1}\rho_{\mathrm{n}-1}\rho_{\mathrm{n}}$,

where $\alpha, \rho_{\mathrm{i}}, \sigma_{\mathrm{i}} \in \mathrm{V}\mathcal{B}_{\mathrm{n}}$, $\rho_{\mathrm{n}}, \sigma_{\mathrm{n}} \in \mathrm{V}\mathcal{B}_{\mathrm{n}+1}$ and ℓ is a natural embedding of $\mathrm{V}\mathcal{B}_{\mathrm{n}}$ to $\mathrm{V}\mathcal{B}_{\mathrm{n}+1}$.

For a virtual braid $\alpha \in \mathrm{V}\mathcal{B}_\mathrm{n}$ denote $\mathrm{F}(\alpha)$ polynomial $\det(\mathrm{I}_\mathrm{n} - \phi(\alpha)) \in \mathbb{Z}[\mathrm{c}^{\pm 1}]$.

Theorem 2

Let $\alpha \in \mathrm{V}\mathcal{B}_\mathrm{n}$. For the Kauffman-Lambropoulou move

$$\alpha \to \ell(\alpha)\sigma_\mathrm{n}^{-1}$$

we have

$$\mathrm{F}(\alpha) = \left(-\frac{1}{\mathrm{c}^2}\right)\mathrm{F}(\ell(\alpha)\sigma_\mathrm{n}^{-1}).$$

For all other Kauffman-Lambropoulou moves $\mathrm{F}(\alpha)$ keeps invariant.

Corollary

Let $\alpha_1 \in \mathrm{V}\mathcal{B}_\mathrm{n}$ and $\alpha_2 \in \mathrm{V}\mathcal{B}_\mathrm{m}$ correspond to the same virtual link, then $\mathrm{F}(\alpha_1) = (-\frac{1}{\mathrm{c}^2})^\mathrm{k}\,\mathrm{F}(\alpha_2)$ for some $\mathrm{k} \in \mathbb{Z}$.

Example of calculation $\mathrm{F}(\beta)$

Consider $\beta = \sigma_1\rho_1 \in \mathrm{V}\mathcal{B}_2$.

$$\phi(\sigma_1\rho_1) = \begin{pmatrix} 0 & \frac{1}{c} \\ \frac{1}{c} & \frac{1+c^2}{c^2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{c} & 0 \\ \frac{1+c^2}{c^2} & \frac{1}{c} \end{pmatrix},$$

$$\mathrm{F}(\sigma_1\rho_1) = \det \begin{pmatrix} -\frac{1}{c}+1 & 0 \\ -\frac{1+c^2}{c^2} & -\frac{1}{c}+1 \end{pmatrix} = \frac{(c-1)^2}{c^2} = \frac{1}{c^2} - \frac{2}{c} + 1.$$

Theorem (T. Kadokami)

Let $\alpha = \prod_{i=1}^{m} \sigma_1^{q_i} \rho_1$, $\beta = \prod_{i=1}^{k} \sigma_1^{p_i} \rho_1$, for some $k, l, q_i, p_i \in \mathbb{Z}$ such that $l, k \geq 1$ and $q_i, p_i \neq 0$. If α and β correspond to the same virtual link then α and β are conjugated in $V\mathcal{B}_2$.

Let $\varkappa(d)$ be a number of vertical lines in diagram $d \in R_n$, f – some function, defined on integers. Define linear map $tr_f : R_n \to \mathbb{R}$ by following equality

$$tr_f(d) = f(\varkappa(d)).$$

Notice

Function $\varkappa : \mathbb{C}R_n \to \mathbb{N}$ is commutative, so tr_f is commutative too.

Let $\mathrm{t} \in \mathbb{C}$ be a complex variable, define linear mapping $\partial : \mathbb{C}\mathrm{R}_2 \to \mathbb{C}\mathrm{R}_2$ assuming that:

$$\partial(\mathrm{d}_1) = \partial(\mathrm{d}_6) = \partial(\mathrm{d}_7) = 0,$$
$$\partial(\mathrm{d}_2) = \mathrm{t}(\mathrm{d}_3 - \mathrm{d}_4) = -\partial(\mathrm{d}_5),$$
$$\partial(\mathrm{d}_3) = \mathrm{t}(\mathrm{d}_2 - \mathrm{d}_5) = -\partial(\mathrm{d}_4).$$

Theorem 3

Mapping $\partial : \mathbb{C}\mathrm{R}_2 \to \mathbb{C}\mathrm{R}_2$ is a derivation on $\mathbb{C}\mathrm{R}_2$, i.e. it satisfies the Leibniz relation

$$\partial(\mathrm{D}_1\mathrm{D}_2) = \partial(\mathrm{D}_1)\,\mathrm{D}_2 + \mathrm{D}_1\,\partial(\mathrm{D}_2).$$

for any $\mathrm{D}_1, \mathrm{D}_2 \in \mathbb{C}\mathrm{R}_2$.

Lemma

Let F – commutative linear function on $\mathbb{C}\mathrm{R}_2$, then composition $\mathrm{F} \circ \partial$ is commutative.

For virtual braid $\beta \in \mathrm{V}\mathcal{B}_2$ and integer $\mathrm{m} \in \mathbb{Z}$ associate the value $\mathrm{T_f} \circ \partial^{\mathrm{m}}(\beta) = \mathrm{tr_f}(\partial^{\mathrm{m}}(\psi(\beta))$.

Theorem 4

Let $\alpha, \beta \in \mathrm{V}\mathcal{B}_2$ be braids satisfying conditions of Kadokami theorem, then for any integer $\mathrm{m} \geq 0$ and any function f we have

$$\mathrm{T_f} \circ \partial^{\mathrm{m}}(\beta) = \mathrm{T_f} \circ \partial^{\mathrm{m}}(\alpha).$$

Example 2

Consider values $\mathrm{T_f}$ and $\mathrm{T_f} \circ \partial$ with $\mathrm{f}(\varkappa) = \varkappa$. It is easy to see, that $\beta_1 = \sigma_1^3\rho_1\sigma_1^2\rho_1\sigma_1\rho_1$ and $\beta_2 = \sigma_1^3\rho_1\sigma_1\rho_1\sigma_1^2\rho_1$ are not conjugated in $\mathrm{V}\mathcal{B}_2$. We have

$$\mathrm{T_f}(\beta_1) = \mathrm{T_f}(\beta_2),$$

but

$$\mathrm{T_f} \circ \partial(\beta_1) \neq \mathrm{T_f} \circ \partial(\beta_2).$$

Thus, the derivation ∂ allows us to distinguish more virtual links.

Thank you for attention!