# Doodles on Surfaces

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This research is a joint work with

Andrew Bartholomew

Roger Fenn

Naoko Kamada

§1. Doodles

§2. Doodles v.s. Virtual doodles

§3. Minimal diagrams of doodles

§4. Doodles and identities among commutators

[BFKK]: A. Bartholomew, R. Fenn, N. Kamada, S. Kamada
Doodles on surfaces I: An introduction to their basic properties, arXiv: 1612.08473

## §1 Doodles

- Roger Fenn & Paul Taylor (1979)
  Doodles were introduced, but were restricted to embedded circles in the 2-sphere

- Mikhail Khovanov (1997)
  extended the idea to immersed circles in the 2-sphere

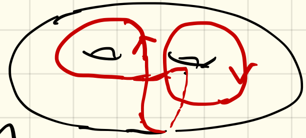
- BFKK (ArXiv: 1612.08473v1)
  We further extend the range of doodles to any closed orientable surfaces

$\Sigma$ : a closed oriented surface

A <u>doodle representative</u> is a generic immersion

$f : \coprod_i S^1_i \longrightarrow \Sigma$

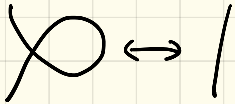
and its image is a <u>doodle diagram</u>.



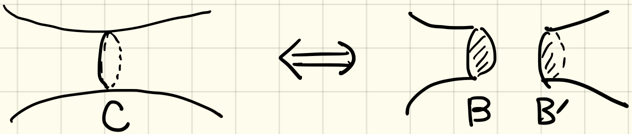
- Equivalence

(1) Homeomorphic equiv

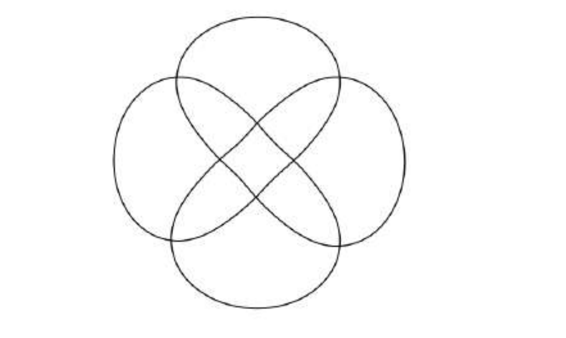
(2) R1 and R2



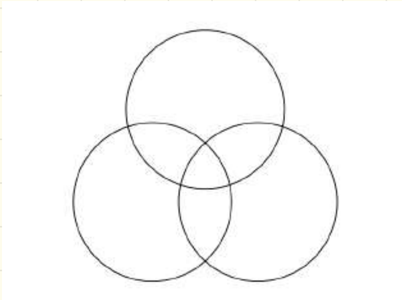
(3) Surface surgery disjoint from the diagram



A **doodle** is an equivalence class of a diagram

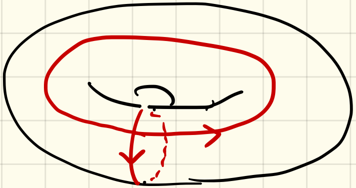


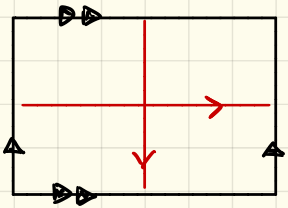
The poppy



The borromean doodle

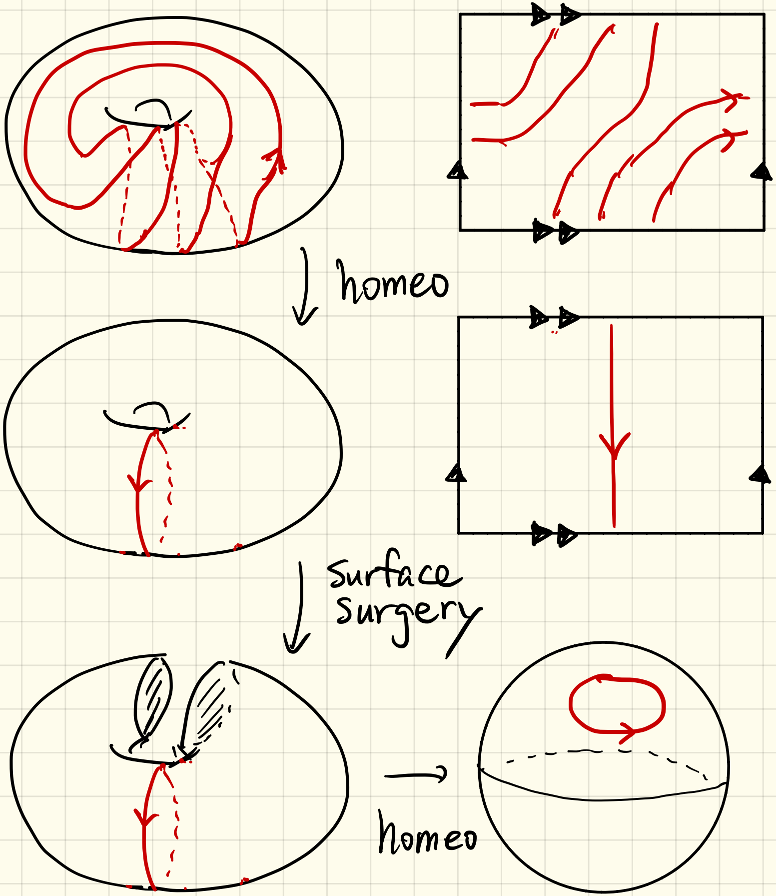
A doodle is **planar** if there is a diagram on $S^2$.





← A doodle diagram on a torus.

The doodle is not planar!

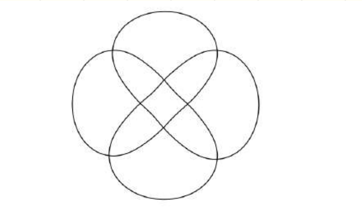


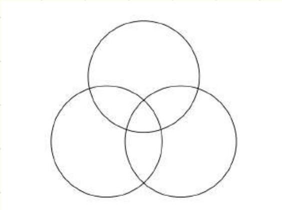

This is the trivial doodle

- Genus

A <u>genus</u> of a doodle is the minimum genus of all surfaces on which there is a diagram representing the doodle.

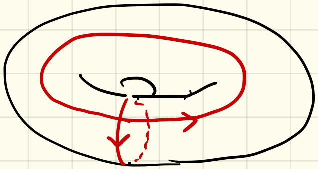
$g=0$



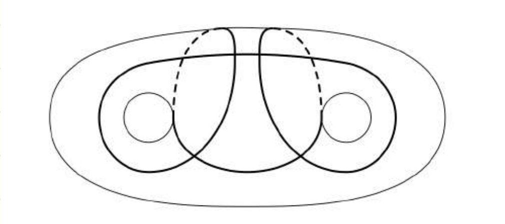


(genus 0 $\Leftrightarrow$ planar)

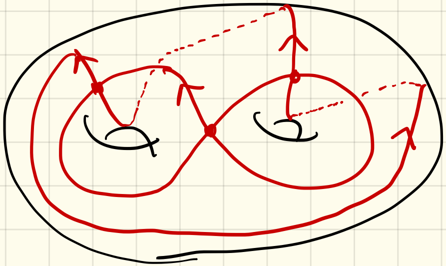
$g=1$



$g=2$



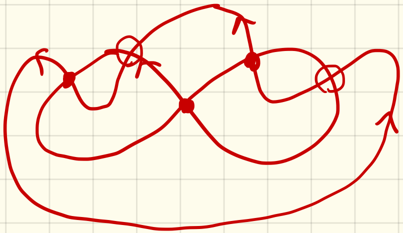
## §2 Doodles v.s. Virtual doodles



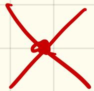
a doodle diagram on a surface

$\rightsquigarrow$ projection

$\leftsquigarrow$ surface realization

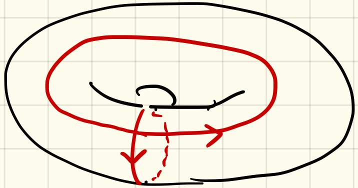


a virtual doodle diagram on $\mathbb{R}^2$



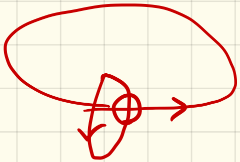
real crossing

virtual crossing

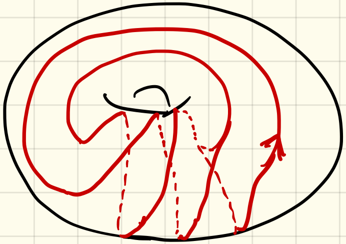


proj $\rightsquigarrow$

$\leftsquigarrow$ surface realization

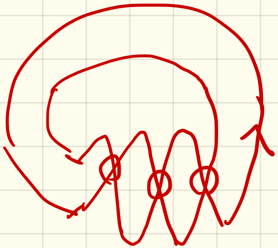


$\not\simeq$

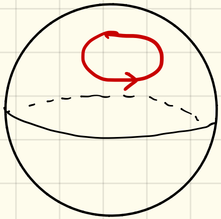


proj $\rightsquigarrow$

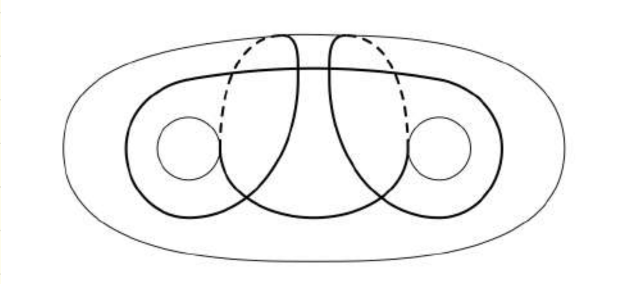
$\leftsquigarrow$ surface realization



$\wr$ equiv

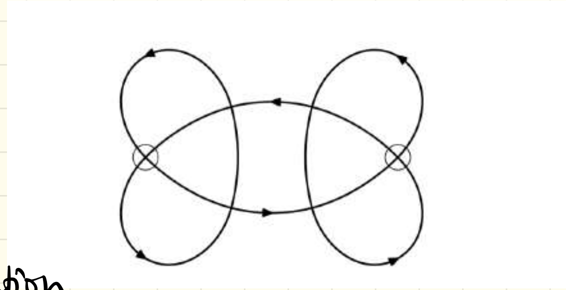


$\wr$ equiv



$\overset{\text{proj}}{\rightsquigarrow}$

$\leftsquigarrow$
surface realization



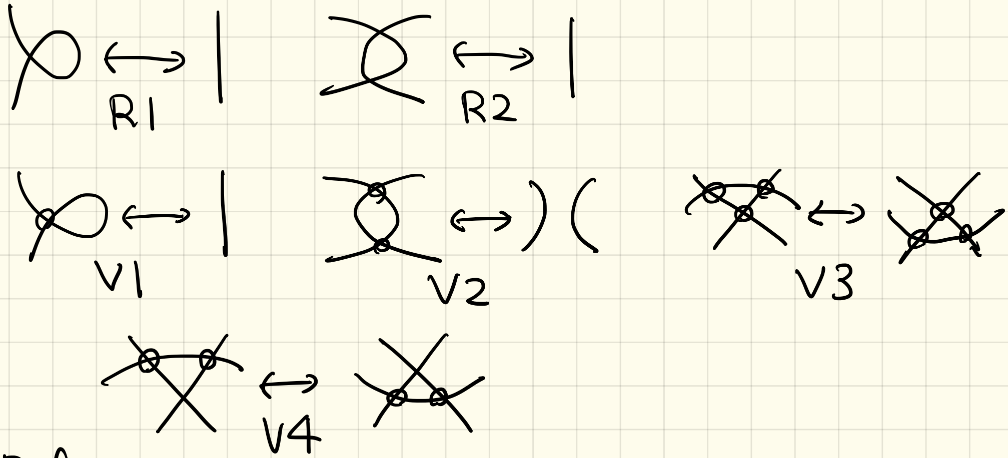
a doodle diagram on a genus-2 surface

Kishino's virtual diagram

**Theorem** (BFKK)

$$\{\text{doodles}\} = \{\text{doodle diagrams on surfaces}\}/\sim \;\overset{1:1}{\longleftrightarrow}\; \{\text{virtual doodles (on } \mathbb{R}^2)\} = \{\text{virtual doodle diagrams on } \mathbb{R}^2\}/\sim$$

- Equivalence for virtual doodle diagrams



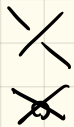

**Definition**

$$\{\text{virtual doodles}\} := \left\{\begin{array}{l}\text{virtual doodle}\\ \text{diagrams}\end{array}\right\} \Big/ \sim$$

R1, R2
V1 ~ V4

**Remark**

$$\{\text{virtual links}\} := \left\{\begin{array}{l}\text{virtual link}\\ \text{diagrams}\end{array}\right\} \Big/ \begin{array}{l}\text{R1, R2, R3}\\ \text{V1} \sim \text{V4}\end{array}$$



# Idea of the proof of Theorem

Theorem (Naoko Kamada–K, Carter–K–Saito)

{virtual links} ⟷ {abstract links} ⟷ {stable equiv. classes of links in thickened surfaces}

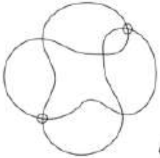
$$\left\{\begin{matrix}\text{virtual link}\\ \text{diagrams on } \mathbb{R}^2\end{matrix}\right\} \Big/ \begin{matrix}R1, R2, R3\\ V1, V2, V3, V4\end{matrix} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{abstract}\\ \text{diagrams}\end{matrix}\right\} \Big/ R1, R2, R3 \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{link}\\ \text{diagrams}\end{matrix}\right\} \Big/ \begin{matrix}R1, R2, R3, \text{homeo},\\ \text{surface surgery}\end{matrix}$$

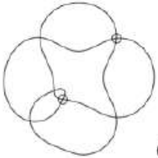
{virtual doodles} ⟷ {abstract doodles} ⟷ {doodles}

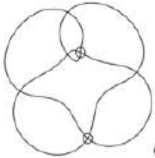
$$\left\{\begin{matrix}\text{virtual doodle}\\ \text{diagrams on } \mathbb{R}^2\end{matrix}\right\} \Big/ \begin{matrix}R1, R2\\ V1, V2, V3, V4\end{matrix} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{abstract}\\ \text{doodle}\\ \text{diagrams}\end{matrix}\right\} \Big/ R1, R2 \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{doodle}\\ \text{diagrams}\end{matrix}\right\} \Big/ \begin{matrix}R1, R2, \text{homeo},\\ \text{surface surgery}\end{matrix}$$

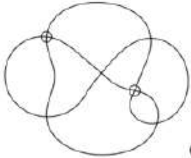
$\square$

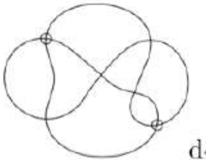
http://www.layer8.co.uk/maths/doodles.

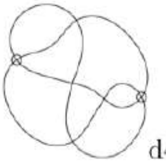

d3.1

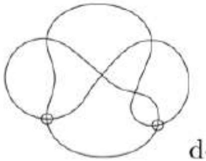

d4.1

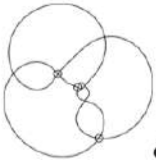

d4.2

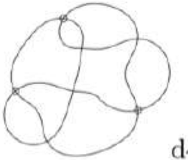

d4.3

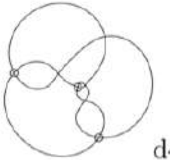

d4.4

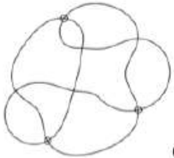

d4.5

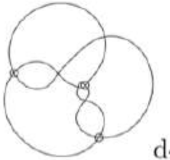

d4.6

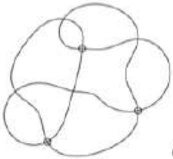

d4.7

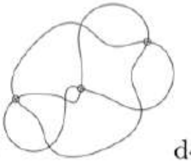

d4.8

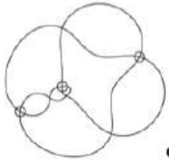

d4.9

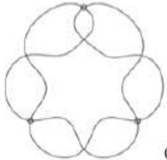

d4.10

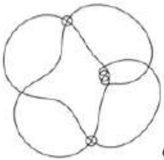

d4.11

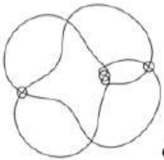

d4.12

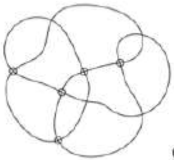

d4.13

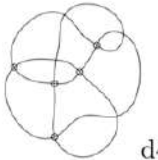

d4.14

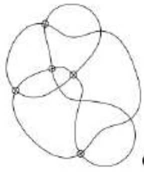

d4.15


d4.16


d4.17


d4.18

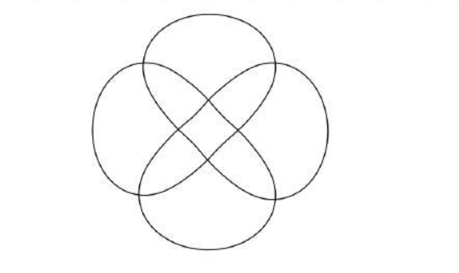

d4.19

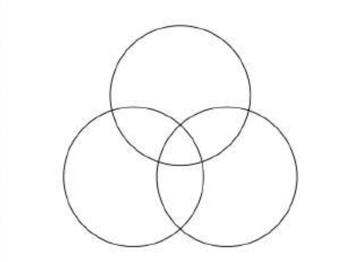

d4.20

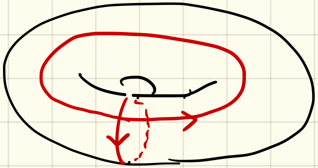
## §3 Minimal diagrams of doodles

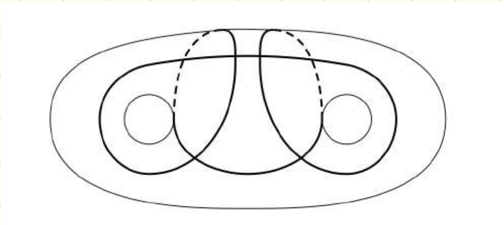
**Definition**

A doodle diagram $D$ on a surface $\Sigma$ is <u>minimal</u> if the interior of every region is simply connected and there are no monogones and bigons.







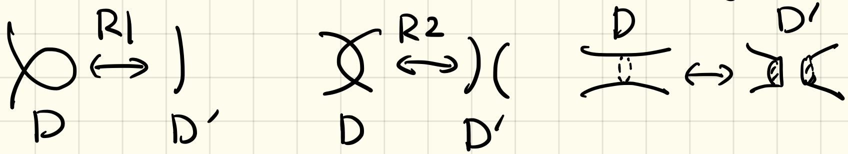


**Theorem** (BFKK)

If two minimal diagrams represent the same doodle, then they are homeomorphic.

**Idea of the proof**

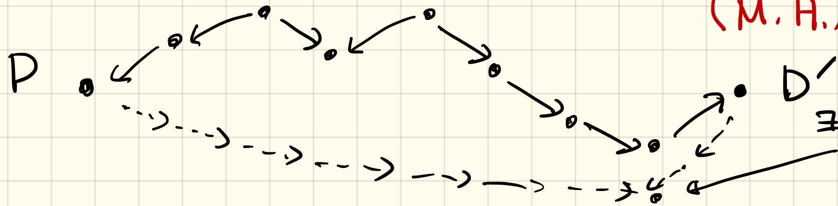
$[D]$ : homeo class of a doodle diagram $D$




$[D] > [D'] \Longleftrightarrow \begin{cases} [D] \searrow [D'] \\ [D'] \swarrow [D] \end{cases}$

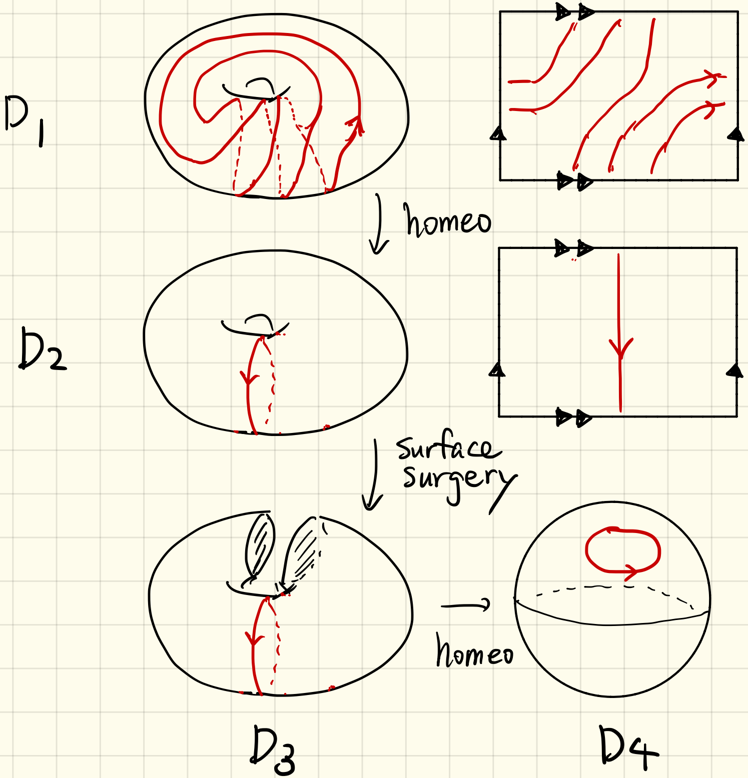
The set of homeo classes $[D]$ with $\searrow, \nearrow$ (level) satisfies conditions of proof reduction

(M. H. A. Newman, Ann of Math 1942)




∃ unique root!

$\square$




**Fact**

$D$ is minimal

$\Longleftrightarrow$ $[D]$ is minimal in $(\mathcal{G}, \searrow, \nearrow)$

$$[D_1] = [D_2] \searrow [D_3] = [D_4]$$

not minimal

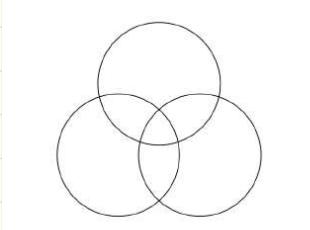
minimal

## Theorem (BFKK)

Consider minimal doodles on $S^2$, so $g=0$.

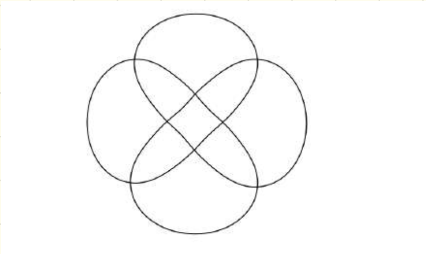
(1) They have at least 6 crossings.

(2) There is only one with 6 rossings: the Borromean doodle.



(3) There are none with 7 crossings.

(4) There is only one with 8 crossings: the poppy.

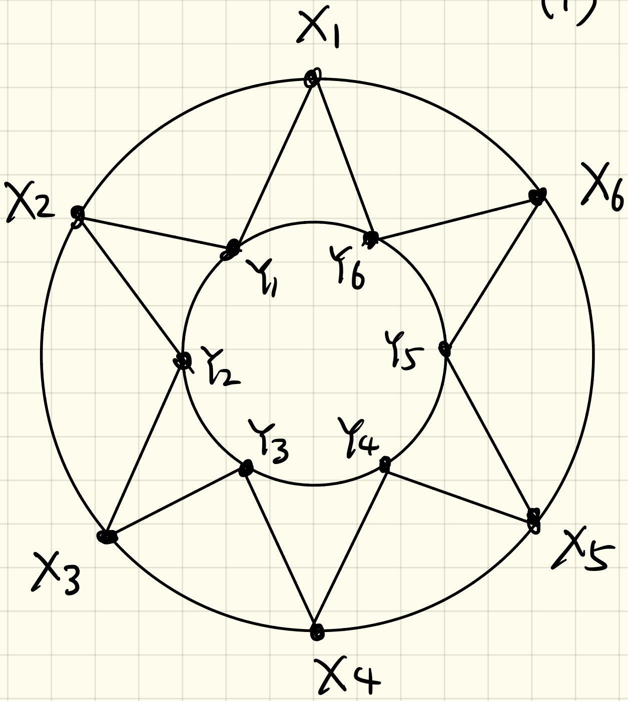


## Idea of Proof

Euler characteristic

$V-E+F=2$

# Infinite sequences of minimal doodles with $g=0$




$B_6$

(1) $B_n$ : the generalized Borromean doodle $(n \geqq 3)$

$B_3$ is the Borromean

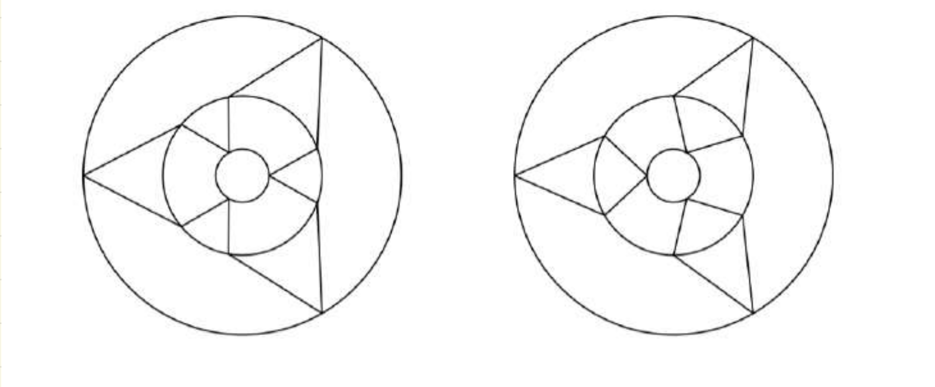
$B_4$ is the poppy

# of components

is $\begin{cases} 3 & (n \equiv 0 \mod 3) \\ 1 & (n \not\equiv 0 \qquad ) \end{cases}$

(2) $C_n'$ : Gyro

(3) $C_n''$ : Ortho



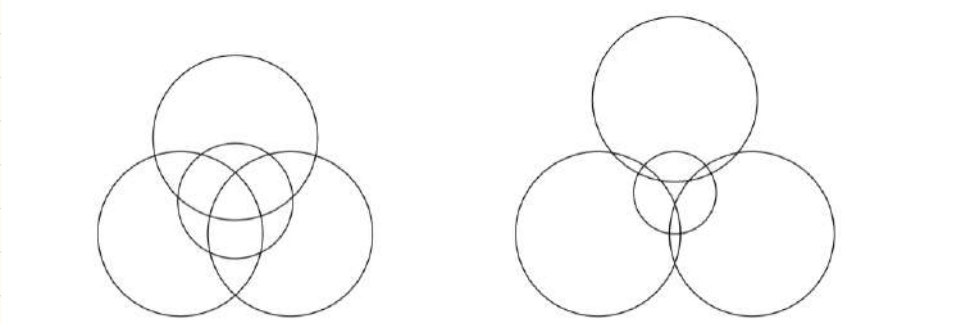
$C_3'$ $\qquad$ $C_3''$

$\#$ of comps of $C_n'$

$$= \begin{cases} 4 & (n \equiv 0 \mod 3) \\ 2 & \end{cases}$$

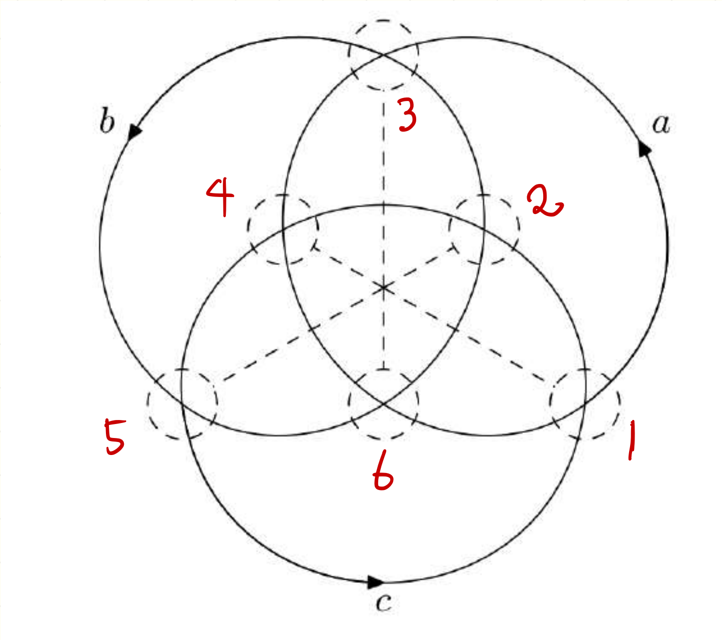
$\#$ of comps of $C_n''$

$= n+1$



$C_3'$ and $C_3''$ with circular components

# §4 Doodles and identities among commutators




① $b^{-1}a^{-1}c^{-1}acb$
$(= [a,c]^b)$

② $b^{-1}c^{-1}bc \quad (= [b,c])$

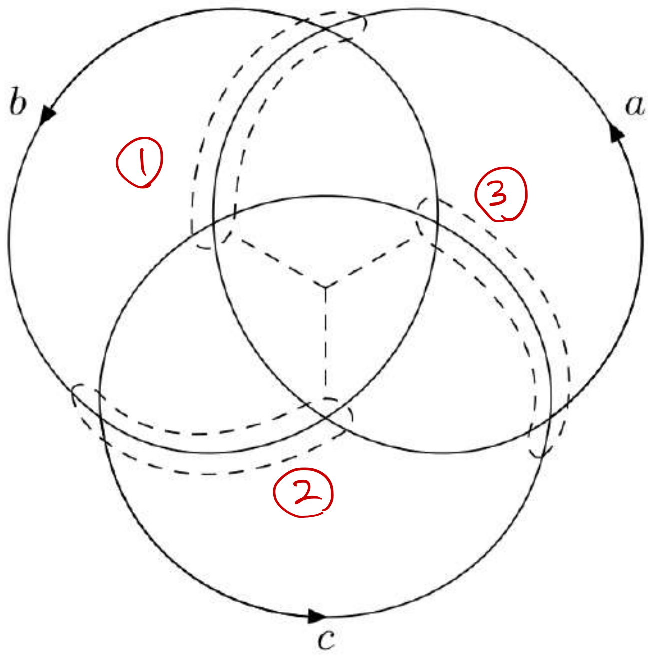
③ $[b,a]^c$

④ $[c,a]$

⑤ $[c,b]^a$

⑥ $[a,b]$

$$[a,c]^b \cdot [b,c] \cdot [b,a]^c \cdot [c,a] \cdot [a,b]^a \cdot [a,b] \equiv 1$$




① $c^{-1}b^{-1}a^{-1}bca$
$(= [bc, a])$

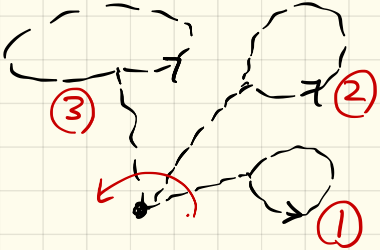
② $a^{-1}c^{-1}b^{-1}cab$
$(= [ca, b])$

③ $b^{-1}a^{-1}c^{-1}abc$
$(= [ab, c])$

$$[bc, a]\cdot[ca, b]\cdot[ab, c] \equiv 1$$

4 A doodle diagram + A noose system

$\rightsquigarrow$ An identity among commutators (commutator identity)

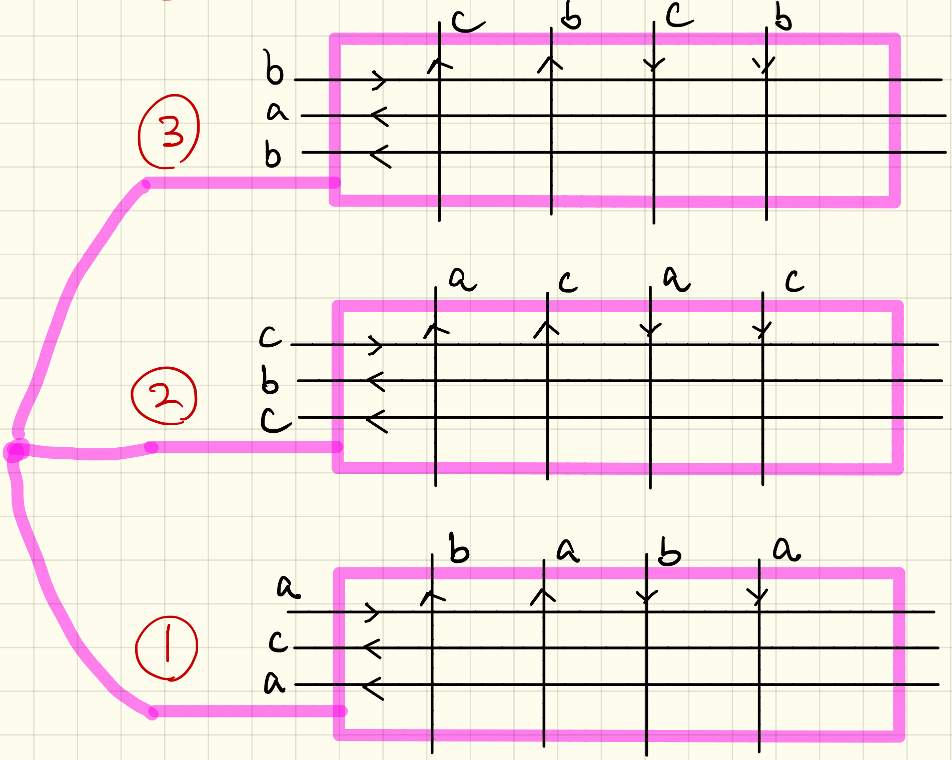


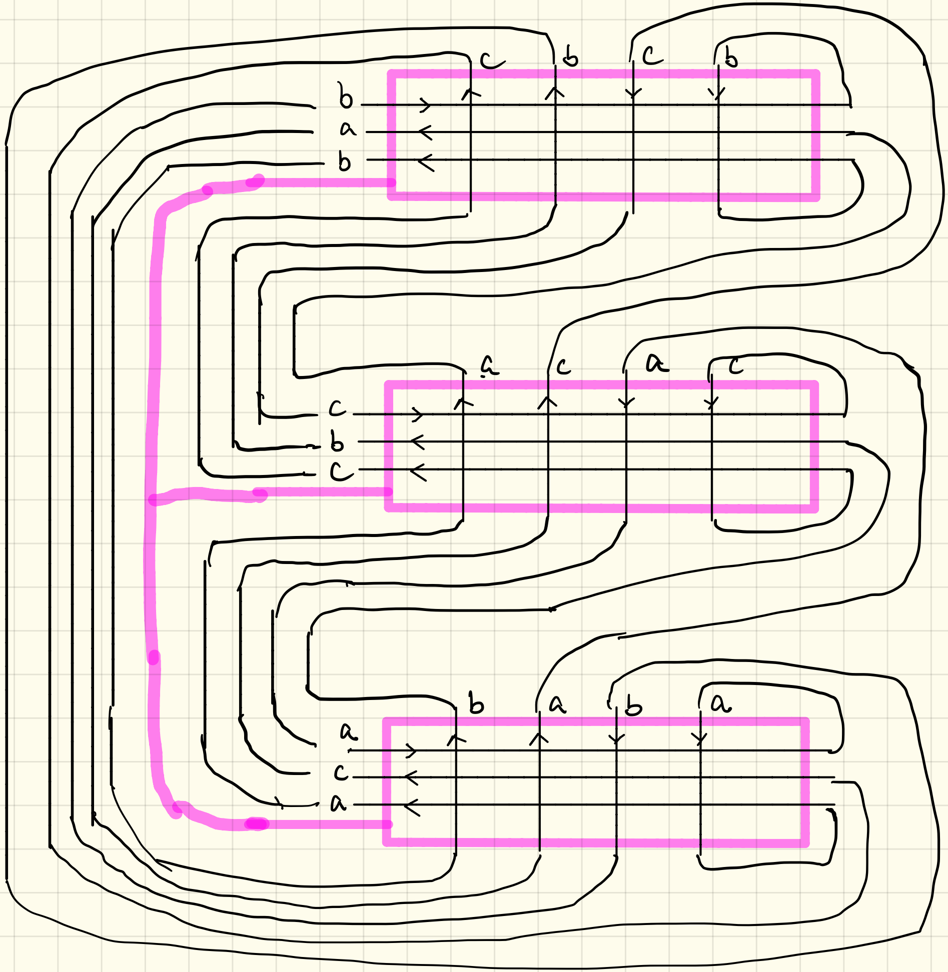

Conversely, for any commutator identity in the free group, there is a doodle diagram and a noose system which yield the identity.

# The Hall-Witt identity

$$[[a,b],c^a]\cdot[[c,a],b^c]\cdot[[b,c],a^b]\equiv 1$$

① ② ③



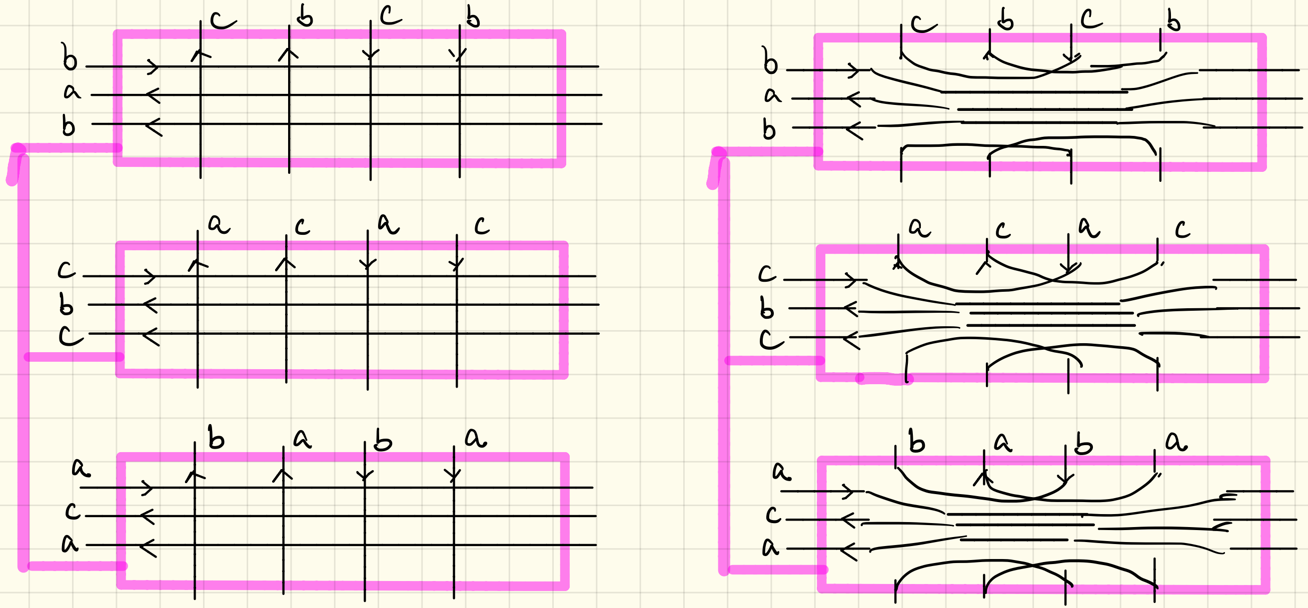


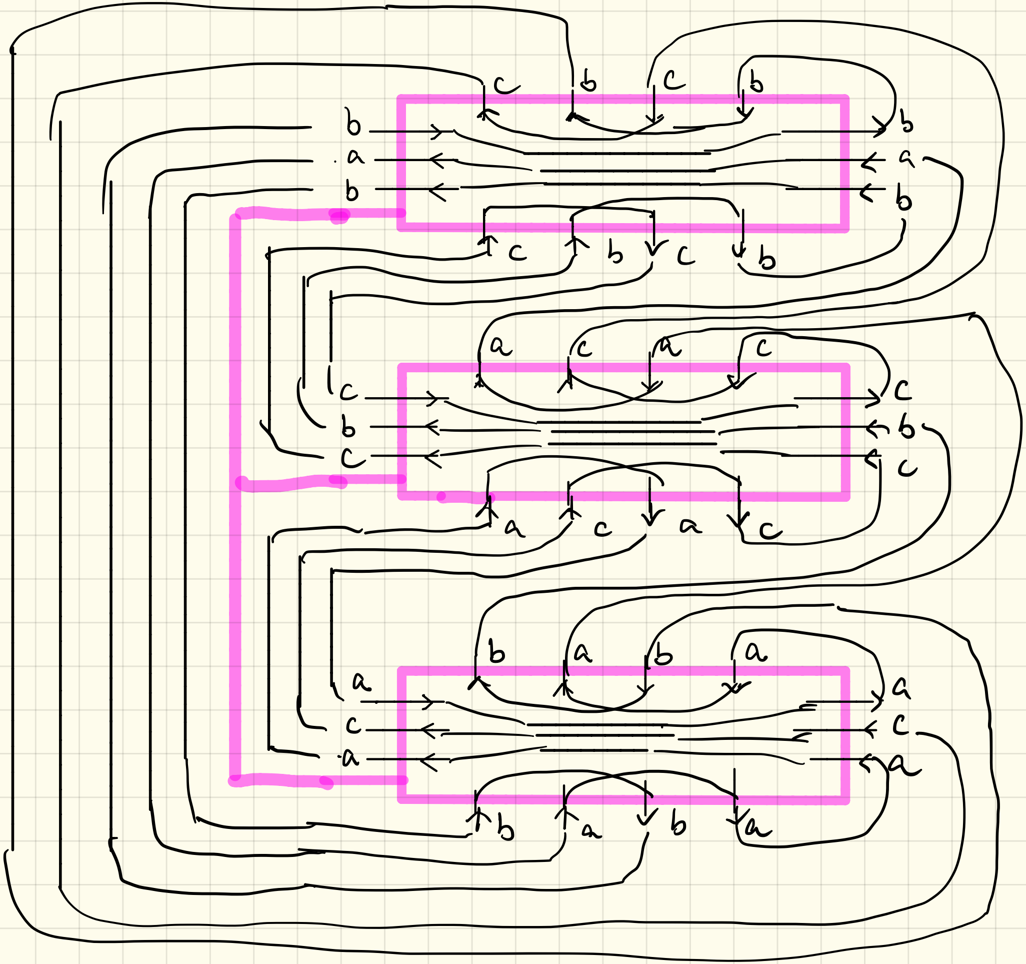

There are 36 crossings.

# The Hall-Witt identity

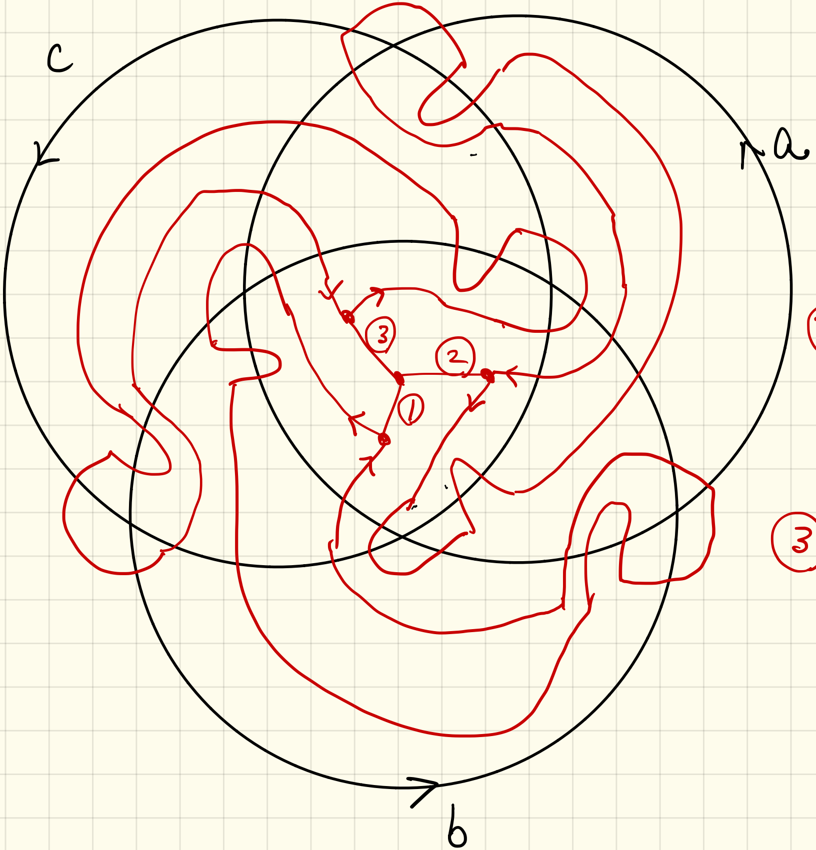
$$[[a,b],c^a]\cdot[[c,a],b^c]\cdot[[b,c],a^b]\equiv 1$$

① ② ③






D has 6 crossings
and minimal,
thus it is
the Borromean
doodle!




① $[b,a]\,a^{-1}c^{-1}a\,[a,b]\,a^{-1}ca$
$= [[a,b], c^a]$

② $[a,c]\,c^{-1}b^{-1}c\,[c,a]\,c^{-1}bc$
$= [[c,a], b^c]$

③ $[c,b]\,b^{-1}a^{-1}b\,[b,c]\,b^{-1}ab$
$= [[b,c], a^b]$

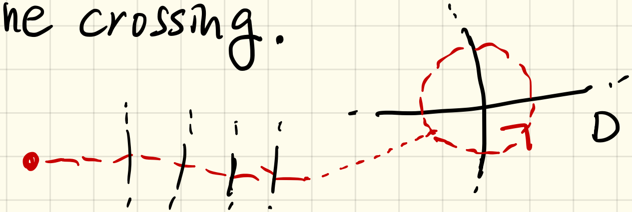
$$\underset{①}{[[a,b], c^a]} \cdot \underset{②}{[[c,a], b^c]} \cdot \underset{③}{[[b,c], a^b]} \equiv 1$$

$S$: a non-empty subset of the free group $F$

<u>Def</u> A commutator identity $[a_1, b_1]^{w_1} \cdots [a_n, b_n]^{w_n} \equiv 1$ is <u>$S$-proper</u> if $\begin{cases} a_i, b_i \in S \amalg S^{-1} \\ w_i \in \mathrm{Word}(S \amalg S^{-1}) \end{cases}$

<u>Def</u> A doodle diagram $D$ is <u>$S$-colored</u> or <u>$S$-partitioned</u> when each component is labeled with an element of $S$.

<u>Def</u> A noose system for $D$ is <u>proper</u> if every noose surround one crossing.



(1) An $S$-colored doodle diagram on $S^2$

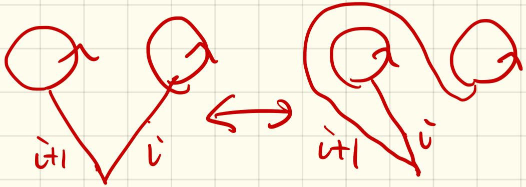
+ a proper noose system

$\rightsquigarrow$ An $S$-proper commutator identity.

Conversely, for any $S$-proper commutator identity in the free group, there is a $S$-colored doodle diagram on $S^2$ and a proper noose system which yield the commutator identity.

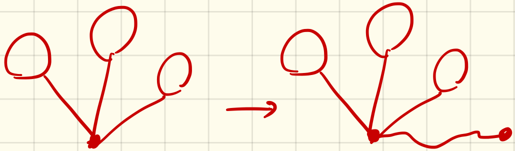
(In the proper case)

① change noose systems $\Longleftrightarrow$ braid group action



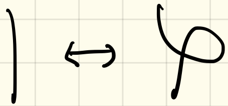

$$[a_i, b_i]^{w_i} \cdot [a_{i+1}, b_{i+1}]^{w_{i+1}} \longleftrightarrow [a_{i+1}, b_{i+1}]^{w_{i+1}} \cdot [a_i, b_i]^{w_i [a_{i+1}, b_{i+1}]^{w_{i+1}}}$$

② change base points $\Longleftrightarrow$ simultaneous conjugation (global conjugation)

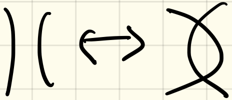


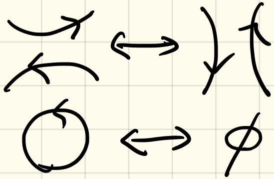
$$\prod [a_i, b_i]^{w_i} \longleftrightarrow \prod [a_i, b_i]^{w_i g}$$

- R1-move $\Longleftrightarrow$ Insertion/deletion of a trivial commutator $[a,a]^w$



- R2-move $\Longleftrightarrow$ Insertion/deletion of a cancelling pair $[a,b]^w \cdot [b,a]^w$



(  $\Longleftrightarrow$ Commutator identities are preserved. )

$$\left\{ \begin{array}{l} \text{S-colored doodles} \\ \text{with } g=0 \end{array} \right\} \xrightarrow{\text{onto}} \left\{ \begin{array}{l} \text{S-proper commutator} \\ \text{identities} \end{array} \right\} \Big/ \sim$$

$\downarrow$ proj.

$\{ \text{———} \} \Big/ \asymp \leftrightarrow )(\,,\ \circ \leftrightarrow \phi \xrightarrow{1:1}$

- A doodle diagram with $g=0$
  $\rightsquigarrow$ commutator identity with RHS $=1$

  A doodle diagram with genus $g$
  $\rightsquigarrow$ commutator identity with RHS $=$
  commutator product of length $g$.

- The idea of doodles can be generalized so that we can obtain commutator identities in braid groups and mapping class groups.

  (in progress)