Parities on 2-knots and 2-links

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The Origins and the Goal

Originally, the notion of *parity* for 1-dimensional (virtual) knots was devised by V.O. Mantutov in 2009. Parity theory for 1-knots (classical, virtual, etc.) boils down to the decoration of crossings, that is codimension 1 singularities of the projection.

Parity allows to refine many knot invariants via parity as well as to create new invariants valued in knot diagrams. Due to existence of "picture-valued" (that is, diagram-valued) invariants the following principle holds for virtual knots:

If a diagram is complicated enough, it can be found as a subdiagram in any equivalent diagram.

A natural idea is to do a similar singularities decoration for objects in higher dimensions. In the present talk that is done for two-dimensional knots.

2-Knots

Definition

A 2-knot (resp. an n-component 2-link) is a smooth embedding in general position of a 2-sphere S^2 (resp. disjoint union of n spheres) into $\mathbb{R}^4$ or S^4 up to isotopy.

If one takes S_g instead of S^2, one obtains a *surface knot* (or *surface link* in case of many components).

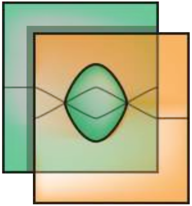
2-Knot Diagrams: in $\mathbb{R}^3$

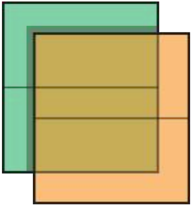
Definition

A $\mathrm{diagram}$ of a 2-knot K in $\mathbb{R}^3$ is a projection in a general position of K in $\mathbb{R}^4$ to a subspace $\mathbb{R}^3$.

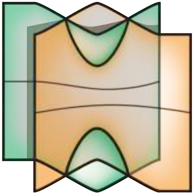
Two diagrams represent the same knot if and only if they can be related by a finite sequence of *Roseman moves*.

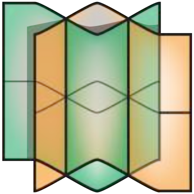
Roseman Move I



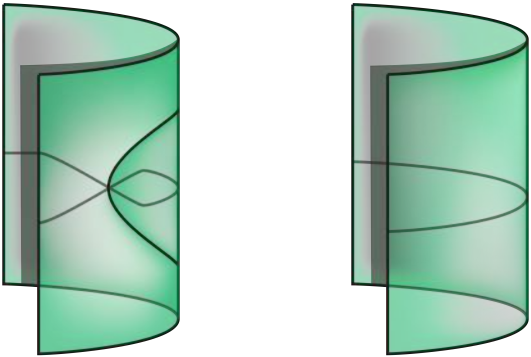


Roseman Move II

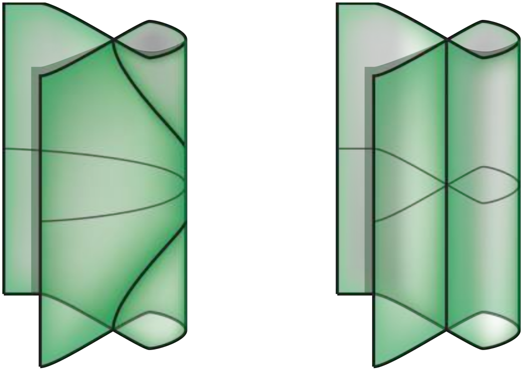




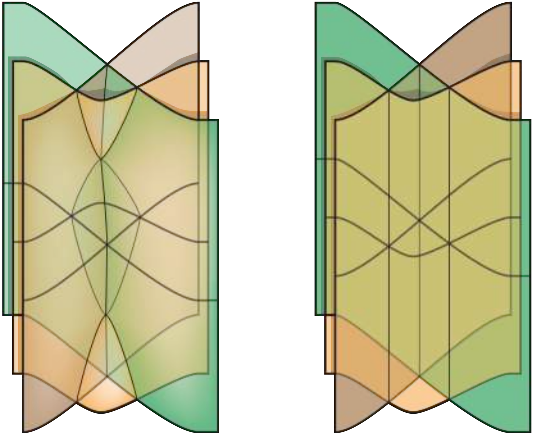
Roseman Move III



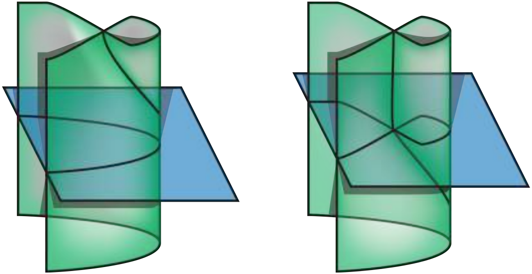
Roseman Move IV



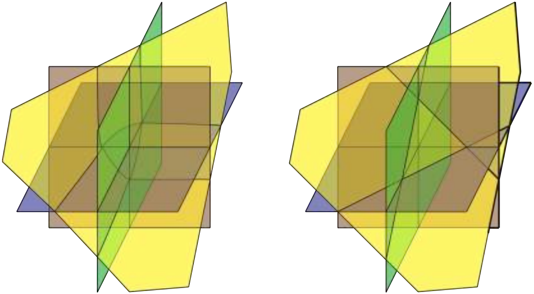
Roseman Move V



Roseman Move VI



Roseman Move VII



Definition

A spherical diagram is a 2-complex consisting of a sphere S and a set D of marked curves on it such that:

1. Every curve is either closed or ends with a cusp, the number of cusps is finite;
2. Every curve of the set D is paired with exactly one curve of that set, one of the paired curves is marked as upper, both curves are oriented (marked with arrows) up to simultaneous orientation change;
3. Two curves ending in the same cusp are paired and both arrows either look towards the cusp or away from it;
4. If two curves intersect, the curves paired with them intersect as well (thus a triple point appears on the sphere S three times).

Roseman moves have natural analogs on spherical diagrams.

Gaussian Parity

Definition

Consider an arbitrary double point x on a double line γ and consider its two preimages x_1, x_2. Connect the point x_1 with the point x_2 by a path $\tilde{\eta}$ so that the behaviour of γ near the endpoints is compatible. $\mathrm{the\ Gaussian\ parity}$ of the double line γ is the parity of the number of intersections between $\tilde{\eta}$ and the preimages of double lines of the knot.

Gaussian Parity and Roseman Moves

The following properties are satisfied:

1. Moves $\mathcal{R}_4$ and $\mathcal{R}_6$ yield that every double line ending in a cusp is even.
2. Two double lines from the second move (located "closely") have the same the Gaussian parity.
3. The double line in the right-hand side of the fourth move is even.
4. The third move creates a double line; it is always even.
5. Among the lines meeting at a triple point there is an even number of odd ones.
6. There are three double lines in the fifth move. There are either two or zero odd among them.

Those properties can be regarded as axioms.

More On Parity Axioms

But those axioms can be rewritten in a more concise and "moves-independent" way:

1. *Continuity axiom.* Parity is constant along double lines.
2. *Selfcrossing axiom.* A double line ending with a cusp is even.
3. *Triple point axiom.* The sum of parities of three double lines meeting in a triple point equals $0 \mod 2$.
4. *Correspondence axiom.* There is a natural bijection between boundary double point on the left and right sides of a Roseman move induced by the bijection of the diagram leaves. The parities of the corresponding points are the same.

Even More on Parity Axioms

There is one more way to construct an equivalent system of axioms:

1. *Continuity axiom.* Parity is constant along double lines.
2. *Loop axiom.* A double line being an edge of a cylinder over a loop is even.
3. *Bigon axiom.* The sum of parities of two double lines being the edges of a cylinder over a bigon equals $0 \mod 2$.
4. *Triangle axiom.* The sum of parities of three double lines being the edges of a cylinder over a triangle equals $0 \mod 2$.
5. *Correspondence axiom.* The parities of the corresponding boundary double points are the same.

Finally we can define *parity* in the following way:

Definition

Let $\mathcal{L}$ be a class of 2-links in $\mathbb{R}^3$, and let A be the set of double line of their diagrams. A mapping $P : A \to \mathbb{Z}_2$ is called parity if it satisfies the axioms 1–5.

Parity Theory Extensions

There are various ways to delve further into parity theories. Just two of them:

1. Weak parity
2. Group-valued parity

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